Exponential and Logarithmic Functions

Algebra 2

Chapter 6

Chapter 6

- This Slideshow was developed to accompany the textbook
  - *Big Ideas Algebra 2*
  - *By Larson, R., Boswell*
  - *2022 K12 (National Geographic/Cengage)*
- Some examples and diagrams are taken from the textbook.

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6-01 Exponent Properties and e (5.2, 6.2)

After this lesson…

- I can simplify expressions with exponents.
- I can simplify expressions involving *e*.
- I can rewrite expressions with *e* as decimals.

6-01 Exponent Properties and e (5.2, 6.2)

- Using Properties of Rational Exponents
  - $x^m \cdot x^n = x^{m+n}$ (Product Property)
  - $(xy)^m = x^m y^m$ (Power of a Product Property)
  - $(x^m)^n = x^{mn}$ (Power of a Power Property)
  - $\frac{x^m}{x^n} = x^{m-n}$ (Quotient Property)
  - $\left(\frac{x}{y}\right)^m = \frac{x^m}{y^m}$ (Power of a Quotient Property)
  - $x^{-m} = \frac{1}{x^m}$ (Negative Exponent Property)

6-01 Exponent Properties and e (5.2, 6.2)

- Simplify the expression. Write your answer using only positive exponents.
- Example 292#3
  - $\left(\frac{3w}{2x}\right)^4$
- Try 292#1
  - $6b^0$

$$\left(\frac{3w}{2x}\right)^4 = \frac{3^4w^4}{2^4x^4} = \frac{81w^4}{16x^4}$$

$$6b^0 = 6 \cdot 1 = 6$$

6-01 Exponent Properties and e (5.2, 6.2)

- e
  - Called the natural base
  - Named after Leonard Euler who discovered it
    - (Pronounced "oil-er")
  - Found by putting really big numbers into $\left(1+\frac{1}{n}\right)^n$ = 2.718281828459…
  - Irrational number like π

6-01 Exponent Properties and e (5.2, 6.2)

- Simplifying natural base expressions
  - Just treat e like a regular variable
- Example (305#5)
  - $(5e^{7x})^4$
- Try 305#3
  - $\frac{11e^9}{22e^{10}}$

$$(5e^{7x})^4 = 5^4 e^{7x\cdot 4} = 625e^{28x}$$

$$\frac{11e^9}{22e^{10}} = \frac{1}{2}e^{9-10} = \frac{1}{2}e^{-1} = \frac{1}{2e}$$

6-01 Exponent Properties and e (5.2, 6.2)

- Evaluate the natural base expressions using your calculator
- Example 305#29
  - Rewrite in the form $y = ab^x$
  - $y = e^{-0.75t}$
- Try 305#31
  - $y = 2e^{0.4t}$

$$y = e^{-0.75t} = (e^{-0.75})^t = 0.472^t$$

$$y = 2e^{0.4t} = 2(e^{0.4})^t = 2(1.492)^t$$

6-01 Exponent Properties and e (5.2, 6.2)

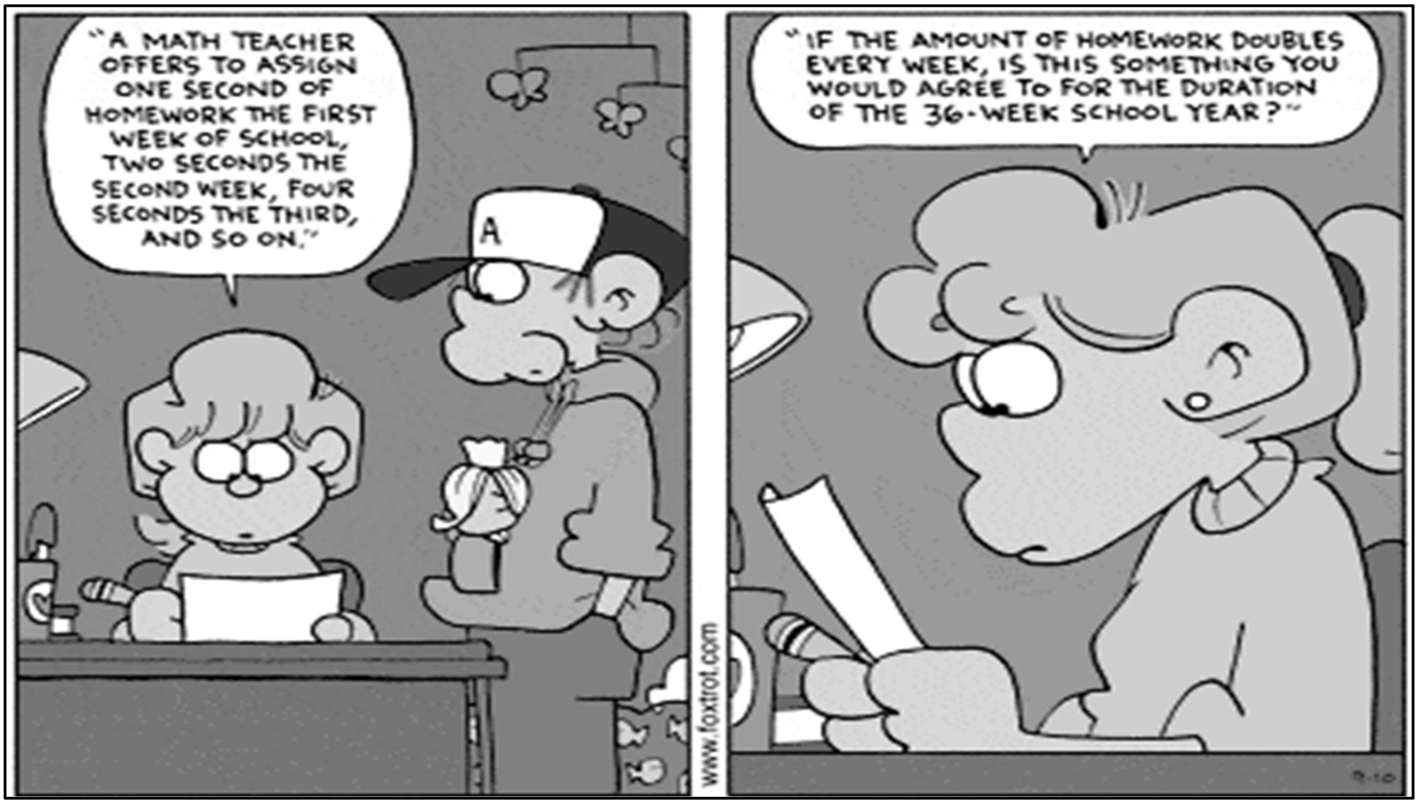
- Assignment (20 total)
  - Properties of Exponents: 292#1-4;
  - Simplifying *e*: 305#1-10 odd;
  - Changing *e* to decimal: 305#25-28 all;
  - Mixed Review: 306#43, 45, 51, 53 (no graph), 55 (no graph)

6-02 Exponential Growth and Decay Functions (6.1)

After this lesson…

- I can identify and graph exponential growth and decay functions.
- I can write exponential growth and decay functions.
- I can solve real-life problems using exponential growth and decay functions.

6-02 Exponential Growth and Decay Functions (6.1)



How much work will be done the last week of school?
Formula is 2^{n-1}
Plug in 36: $2^{36-1} = 3.436\text{x}10^{10}$ seconds → 9544371.769 hours → 397682.157 days → 1088.8 years


I'M A LITTLE WORRIED ABOUT ANSWERING THIS ONE WRONG.
IS THAT MR. WRIGHT? HE'S EVIL.

6-02 Exponential Growth and Decay Functions (6.1)

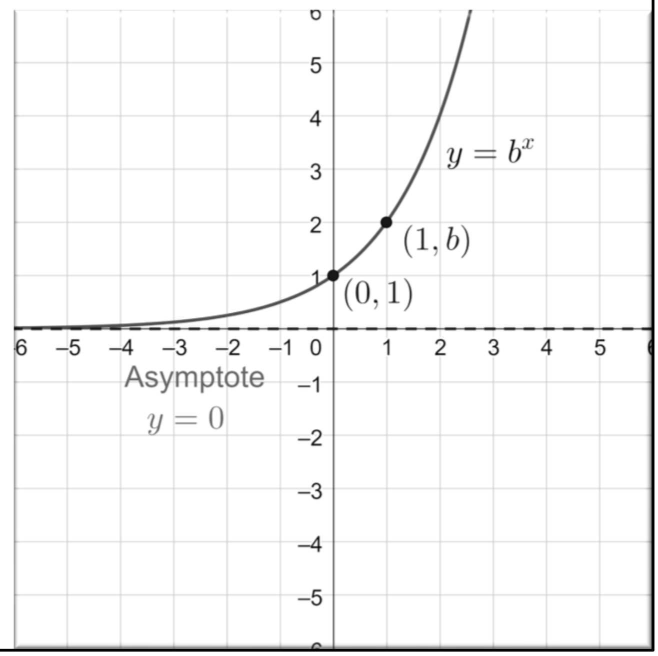
Work with a partner.

Calculate how much time you will spend on your homework the last week of the 36-week school year. You start with 1 second of homework on week one and double the time every week.

2^{35}

6-02 Exponential Growth and Decay Functions (6.1)

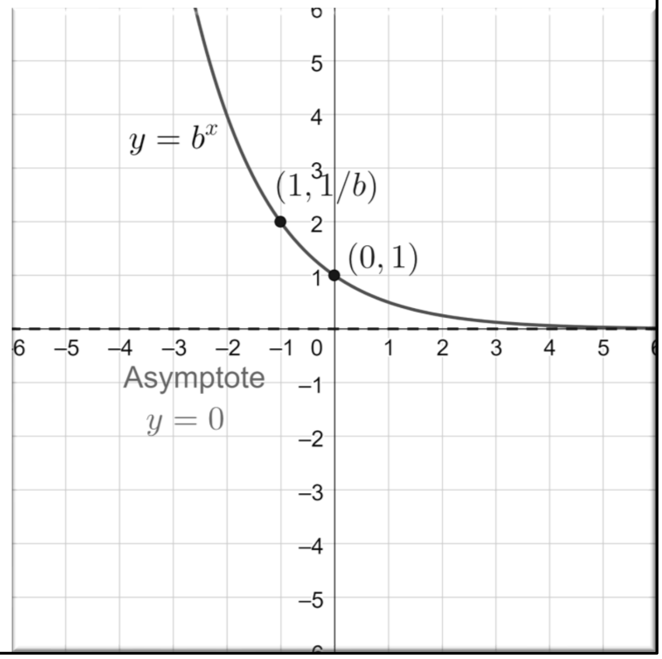
- Exponential Function
  - $y = b^x$
  - Base (b) is a positive number other than 1



- Exponential Growth
  - Always increasing and rate of change is increasing
  - $b > 1$
  - y-intercept is (0, 1)
  - Horizontal asymptote $y = 0$
  - b is the growth factor

6-02 Exponential Growth and Decay Functions (6.1)

- Exponential Decay
  - Always decreasing and rate of change is decreasing
  - 0 < b < 1
  - y-intercept is (0, 1)
  - Horizontal asymptote y = 0
  - b is the decay factor



6-02 Exponential Growth and Decay Functions (6.1)

- Example 298#9
- Determine whether each function represents *exponential growth* or *exponential decay*. Then graph the function.
  $f(x) = \left(\frac{1}{6}\right)^x$

b = 1/6 < 1 exponential decay

6-02 Exponential Growth and Decay Functions (6.1)

- Try 298#11
- Determine whether each function represents *exponential growth* or *exponential decay*. Then graph the function.
  $y = \left(\frac{4}{3}\right)^x$

b = 4/3 > 1 exponential growth

6-02 Exponential Growth and Decay Functions (6.1)

- Exponential <u>Growth</u> Model (word problems)
  - $y = a(1 + r)^t$
    - y = current amount
    - a = initial amount
    - r = growth percent
    - $1 + r$ = growth factor
    - t = time
- Exponential <u>Decay</u> Model (word problems)
  - $y = a(1 - r)^t$
    - y = current amount
    - a = initial amount
    - r = decay percent
    - $1 - r$ = decay factor
    - t = time

6-02 Exponential Growth and Decay Functions (6.1)

- Example: 298#20
  - The population P (in millions) of Peru during a recent decade can be approximated by $P = 28.22(1.01)^t$, where t is the number of years since the beginning of the decade.
  - (a) Determine whether the model represents exponential growth or decay
  - (b) identify the annual percent increase or decrease in population
  - (c) Estimate when the population was about 30 million

a) Base is 1.01 > 1 growth
b) 1.01 = 1 + r → 0.01 = r = 1%
c) $30 = 28.22(1.01)^t$

$$1.063 = 1.01^t$$
$$\log_{1.01} 1.063 = \log_{1.01} 1.01^t$$
$$6.15 = t$$

about 6.2 years since the beginning of the decade

6-02 Exponential Growth and Decay Functions (6.1)

- Try 298#19
- The value of a mountain bike y (in dollars) can be approximated by the model $y = 200(0.65)^t$, where t is the number of years since the bike was purchased.
- (a) Determine whether the model represents exponential growth or decay
- (b) Identify the annual percent increase or decrease
- (c) Estimate when the value of the bike will be \$50

a) Base = 0.65 < 1 decay
b) 0.65 = 1 – r → -0.35 = -r → r = 0.35 = 35%
c) $50 = 200(0.65)^t$

$$0.25 = 0.65^t$$
$$\log_{0.65} 0.25 = \log_{0.65} 0.65^t$$
$$3.22 = t$$

about 3.2 years after it was purchased

6-02 Exponential Growth and Decay Functions (6.1)

- Compound Interest
- $A = P\left(1 + \frac{r}{n}\right)^{nt}$
  - A = amount at time t
  - P = principle (initial amount)
  - r = annual rate
  - n = number of times interest is compounded per year

- Example: 299#39
- Find the balance in the account earning compound interest after 6 years when the principle is $3500. r = 2.16%, compounded quarterly

- Try 299#41

$$A = P\left(1 + \frac{r}{n}\right)^{nt}$$

$$A = 3500\left(1 + \frac{0.0216}{4}\right)^{4(6)} = \$3982.92$$

$$A = 3500\left(1 + \frac{0.0126}{12}\right)^{12(6)} = \$3688.56$$

6-02 Exponential Growth and Decay Functions (6.1)

- Assignment: 20 total
  - Graphing Exponential Growth and Decay: 298#7-15 odd
  - Exponential Growth and Decay Models: 298#19-22, 44
  - Compound Interest: 299#35, 39, 40, 41, 42
  - Mixed Review: 300#53, 54, 55, 61, 63

6-03 Rewrite Exponential as Logarithmic Functions (6.3)

After this lesson…

- I can evaluate logarithms.
- I can rewrite exponential equations as logarithmic equations.
- I can rewrite logarithmic equations as exponential equations.

6-03 Rewrite Exponential as Logarithmic Functions (6.3)

- Logarithms are exponents
  - $\log_b a =$ exponent of b to get a
- Example: 312#13
  - $\log_3 81$
- Try 312#15
  - $\log_3 3$

$$\log_3 81$$
$$3^x = 81$$
$$x = 4$$

Can use trial and error

$$\log_3 3$$
$$3^x = 3$$
$$x = 1$$

6-03 Rewrite Exponential as Logarithmic Functions (6.3)

- Calculator has two logs
  - Common Log: log = $\log_{10}$
  - Natural Log: ln = $\log_e$
- (Some calculators can do log of any base.)
- Example: 312#23
  - $\log 6$
- Try 312#25
  - $\ln \frac{1}{3}$

$$\log 6 = 0.778$$

$$\ln \frac{1}{3} = -1.099$$

6-03 Rewrite Exponential as Logarithmic Functions (6.3)

- Definition of Logarithm with Base *b*
- $\log_b y = x \Leftrightarrow b^x = y$
- Read as "log base *b* of *y* equals *x*"
- Logs = <u>exponents!!</u>
- Logs and exponentials are inverses
  - They undo each other
  - They cancel each other out

$3^2 = 9$
$8^0 = 1$
$5^{-2} = 1/25$

6-03 Rewrite Exponential as Logarithmic Functions (6.3)

- Example: 312#1
  - Rewrite as an exponential
  - $\log_3 9 = 2$
- Try 312#7
  - Rewrite as a log
  - $6^2 = 36$

$$\log_3 9 = 2$$

Base = 3, exponent = 2, other = 9

$$3^2 = 9$$

$$6^2 = 36$$

Base = 6, exponent = 2, other = 36

$$\log_6 36 = 2$$

6-03 Rewrite Exponential as Logarithmic Functions (6.3)

- Simplify log expressions
  - If exponential with base *b* and log with base *b* are inside each other, they cancel
- Example: 312#31
  - $7^{\log_7 x}$
- Try 312#35
  - $\log_3 3^{2x}$

$$7^{\log_7 x} = \cancel{7^{\log_7}}\, x = x$$

$$\log_3 3^{2x} = \cancel{\log_3 3}^{2x} = 2x$$

6-03 Rewrite Exponential as Logarithmic Functions (6.3)

- Assignment (20 total)
  - Evaluate logs: 312#13, 15, 17, 23, 25
  - Rewrite logs as exponentials: 312#1, 3, 5
  - Rewrite exponentials as logs: 312#7, 9, 11
  - Simplify expressions: 312#31, 33, 35, 37
  - Mixed Review: 314#75, 77, 79, 83, 85

6-04 Logarithmic Properties (6.5)

After this lesson…

- I can expand logarithms.
- I can condense logarithms.
- I can evaluate logarithms using the change-of-base formula.

6-04 Logarithmic Properties (6.5)

- Product Property
  - $\log_b uv = \log_b u + \log_b v$

- Quotient Property
  - $\log_b \frac{u}{v} = \log_b u - \log_b v$

- Power Property
  - $\log_b u^n = n \log_b u$

6-04 Logarithmic Properties (6.5)

- Expand logarithms
  - Rewrite as several logs
- Example: 327#13
  - $\log 10x^5$
- Try 327#15
  - $\ln \frac{x}{3y}$

$$\log 10x^5$$

Product property

$$\log 10 + \log x^5$$

Power property

$$\log 10 + 5\log x$$

Simplify

$$1 + 5\log x$$

$$\ln \frac{x}{3y}$$

Quotient and power properties

$$\ln x - \ln 3 - \ln y$$

6-04 Logarithmic Properties (6.5)

- Condense logs
  - Try to write as a single log
- Example: 327#25
  - $\log_5 4 + \frac{1}{3}\log_5 x$
- Try 327#23
  - $6 \ln x + 4 \ln y$

$$\log_5 4 + \frac{1}{3}\log_5 x$$

Power Property

$$\log_5 4 + \log_5 x^{\frac{1}{3}}$$

Product Property

$$\log_5\left(4x^{\frac{1}{3}}\right)$$

Write as radical (because of fractional exponent)

$$\log_5 4\sqrt[3]{x}$$

$$6 \ln x + 4 \ln y$$

Power Property

$$\ln x^6 + \ln y^4$$

Product property

$$\ln x^6 y^4$$

6-04 Logarithmic Properties (6.5)

- Change-of-Base Formula
  - $\log_c u = \frac{\log_b u}{\log_b c}$
  - This lets you evaluate any log on a calculator
- Example: 327#31
  - Evaluate $\log_9 15$
- Try 327#29
  - Evaluate $\log_4 7$

$$\log_9 15 = \frac{\log 15}{\log 9} = 1.232$$

$$\log_4 7 = \frac{\log 7}{\log 4} = 1.404$$

6-04 Logarithmic Properties (6.5)

- Assignment: 20 total
  - Expand logs: 327#11-17 odd
  - Condense logs: 327#21-27 odd
  - Change-of-base formula: 327#29-35 odd
  - Problem Solving: 327#37-38 (Use $L = 10 \log \frac{I}{10^{-12}}$)
  - Mixed Review: 328#46, 47, 51, 57, 59, 61

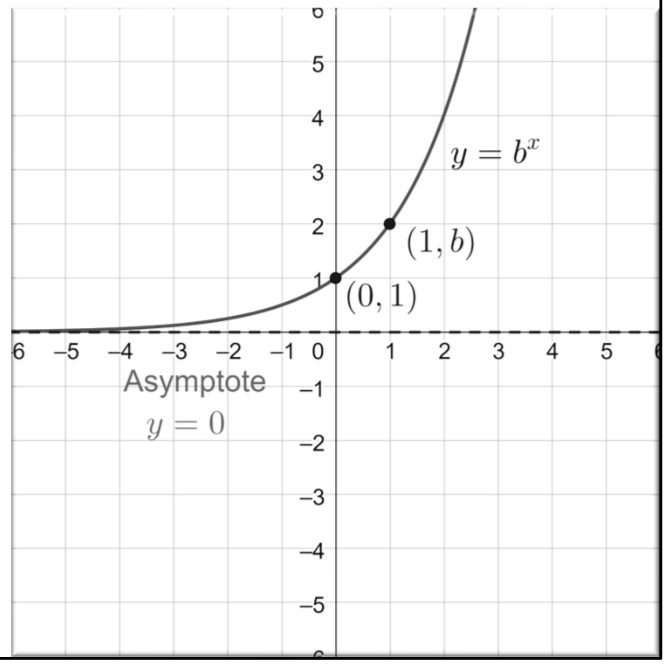
6-05 Graph Exponential and Logarithmic Functions (6.4)

After this lesson…

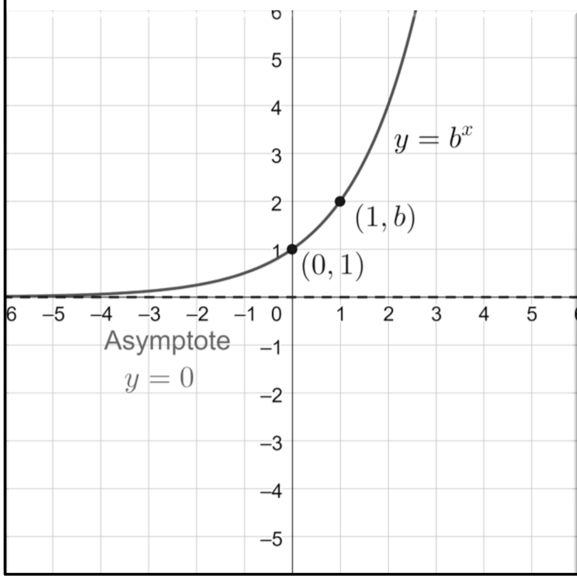
- I can graph exponential functions.
- I can graph logarthmic functions.
- I can find inverses of exponential and logarithmic functions.

6-05 Graph Exponential and Logarithmic Functions (6.4)

- Exponential Function
  - $y = b^x$
  - Base (b) is a positive number other than 1
- In general
  - $y = ab^{cx-h} + k$
  - a is vertical stretch
    - If a is −, reflect over x-axis
  - c is horizontal shrink
    - Shrink by $\frac{1}{c}$
    - If c is −, reflect over y-axis
  - h is horizontal shift
  - k is vertical shift
  - Horizontal asymptote: $y = k$



6-05 Graph Exponential and Logarithmic Functions (6.4)



- Graph Exponential Functions
  - Find and graph the horizontal asymptote
  - Make a table of values
  - Plot points and draw the curve
    - Make sure the curve is near the asymptotes at the edge of the graph

6-05 Graph Exponential and Logarithmic Functions (6.4)

- Example: 320#17
- (a) Describe the transformations. (b) Then graph the function.
  $g(x) = -2^{x-3}$

Transformations: a = -1, h = 3, k = 0
Reflection in x-axis ($-a$) and shift 3 to right
Horizontal asymptote: y = 0

6-05 Graph Exponential and Logarithmic Functions (6.4)

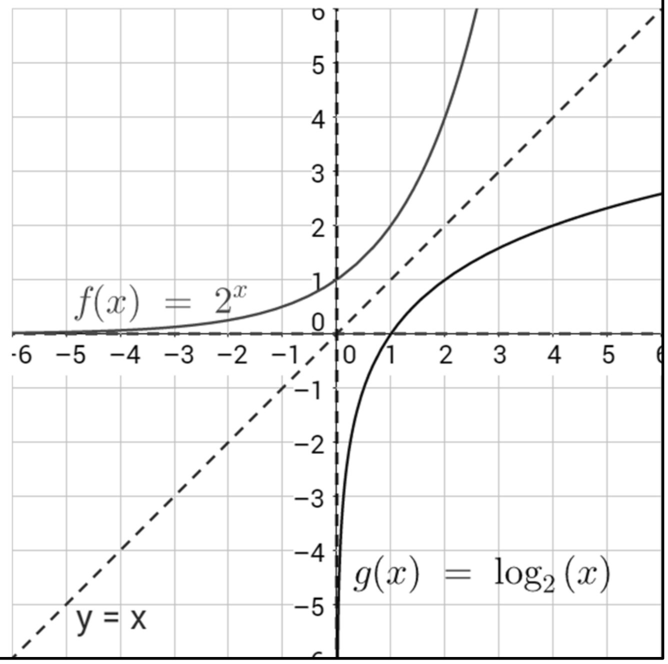
- Try 320#15
- (a) Describe the transformations. (b) Then graph the function.
  $g(x) = e^{2x}$

Transformations: *a* = 1, c = 2, *h* = 0, *k* = 0
Horizontal shrink by factor of 1/2
Horizontal asymptote: *y* = 0

6-05 Graph Exponential and Logarithmic Functions (6.4)

- Logarithmic Function
  - $y = \log_b x$
  - Base (b) is a positive number other than 1

  - Logarithms and exponentials are inverses
  - x and y are switched
  - Graphically, reflected over $y = x$
  - Horizontal asymptote becomes vertical asymptote



6-05 Graph Exponential and Logarithmic Functions (6.4)

- In general
  - $y = a\log_b(cx - h) + k$
  - a is vertical stretch
    - If a is −, reflect over x-axis
  - c is horizontal shrink
    - Shrink by $\frac{1}{c}$
    - If c is −, reflect over y-axis
  - h is horizontal shift
  - k is vertical shift
  - Vertical asymptote: $x = h$
- Graph Logarithmic Functions
  - Find and graph the vertical asymptote
  - Make a table of values
    - You may need to use the change-of-base formula
  - Plot points and draw the curve
    - Make sure the curve is near the asymptotes at the edge of the graph

To put in calculator, you might need to use change-of-base formula

$$y = \log_3 x$$

$$y = \frac{\log x}{\log 3}$$

6-05 Graph Exponential and Logarithmic Functions (6.4)

- Example: 320#27
- (a) Describe the transformations. (b) Then graph the function.
  $g(x) = -\log_{1/5}(x - 7)$

Transformations: a = -1, h = 7, k = 0
Reflection in x-axis ($-a$) and shift 7 to right
Vertical asymptote: x = 7

6-05 Graph Exponential and Logarithmic Functions (6.4)

- Try 320#25
- (a) Describe the transformations. (a) Then graph the function.
  $g(x) = 3\log_4 x - 5$

Transformations: *a* = 3, *h* = 0, *k* = -5
Vertical stretch by factor of 3; vertical shift up 5
Vertical asymptote: *x* = 0

6-05 Graph Exponential and Logarithmic Functions (6.4)

- Find the inverse
  - Isolate log or exponential part
  - Switch *x* and *y*
  - Then rewrite as exponential or log
- Example: 313#47
  - $y = \ln(x - 1)$
- Try 313#51
  - $y = 5^x - 9$

$$y = \ln(x - 1)$$
$$x = \ln(y - 1)$$

Base = e, exponent = x, other = y – 1

$$y - 1 = e^x$$
$$y = e^x + 1$$

$$y = 5^x - 9$$
$$y - 9 = 5^x$$
$$x - 9 = 5^y$$

Base = 5, exponent = y, other x – 9

$$y = \log_5(x - 9)$$

6-05 Graph Exponential and Logarithmic Functions (6.4)

- Assignment: 15 total
  - Graph Exponential Functions: 320#15, 17, 21
  - Graph Logarithmic Functions: 313#57, 59; 320#25, 27
  - Find Inverses: 313#43, 45, 47, 51
  - Mixed Review: 322# 53, 55, 62, 65

6-06 Solve Exponential and Logarithmic Equations (6.6)

After this lesson…

- I can solve exponential equations.
- I can solve logarithmic equations.

6-06 Solve Exponential and Logarithmic Equations (6.6)

- Solving Exponential Equations
  - Method 1) if the bases are equal, then exponents are equal
- Example: 334#3
  - $5^{x-3} = 25^{x-5}$
- Try 334#1
  - $2^{3x+5} = 2^{1-x}$

$$5^{x-3} = 25^{x-5}$$

Write at same base

$$5^{x-3} = 5^{2(x-5)}$$

Since bases are same, exponents are the same

$$x - 3 = 2(x - 5)$$
$$x - 3 = 2x - 10$$
$$-3 = x - 10$$
$$7 = x$$

$$2^{3x+5} = 2^{1-x}$$

Since bases are same, exponents are the same

$$3x + 5 = 1 - x$$
$$4x + 5 = 1$$
$$4x = -4$$
$$x = -1$$

6-06 Solve Exponential and Logarithmic Equations (6.6)

- Solving Exponential Equations
  - Method 2) take log of both sides
- Example: 334#9
  - $5(7)^{5x} = 60$
- Try 334#11
  - $3e^{4x} + 9 = 15$

$$5(7)^{5x} = 60$$
$$7^{5x} = 12$$

Log both side with base 7

$$\log_7 7^{5x} = \log_7 12$$
$$5x = \log_7 12$$
$$x = \frac{\log_7 12}{5} \approx 0.255$$

$$3e^{4x} + 9 = 15$$
$$3e^{4x} = 6$$
$$e^{4x} = 2$$

Log both sides with base e

$$\ln e^{4x} = \ln 2$$
$$4x = \ln 2$$
$$x = \frac{\ln 2}{4} \approx 0.173$$

6-06 Solve Exponential and Logarithmic Equations (6.6)

- Solving Logarithmic Equations
  - Method 1) if the bases are equal, then logs are equal
- Example 334#17
  - $\ln(4x - 12) = \ln x$
- Try 334#19
  - $\log_2(3x - 4) = \log_2 5$

$$\ln(4x - 12) = \ln x$$

Since logs are the same, the stuff in logs are the same

$$4x - 12 = x$$
$$-12 = -3x$$
$$4 = x$$

$$\log_2(3x - 4) = \log_2 5$$

Since logs are the same, the stuff in the logs are the same

$$3x - 4 = 5$$
$$3x = 9$$
$$x = 3$$

6-06 Solve Exponential and Logarithmic Equations (6.6)

- Solving Logarithmic Equations
  - Method 2) exponentiating both sides
    - Make both sides exponents with the base of the log
- Example: 334#21
  - $\log_2(4x + 8) = 5$
- Try 334#22
  - $\log_3(2x + 1) = 2$

$$\log_2(4x + 8) = 5$$

Exponentiate with base 2

$$2^{\log_2(4x+8)} = 2^5$$
$$4x + 8 = 32$$
$$4x = 24$$
$$x = 6$$

$$\log_3(2x + 1) = 2$$

Exponentiate with base 3

$$3^{\log_3(2x+1)} = 3^2$$
$$2x + 1 = 9$$
$$2x = 8$$
$$x = 4$$

6-06 Solve Exponential and Logarithmic Equations (6.6)

- Assignment (20 total)
  - Solve Exponential Equations: 334#1, 3, 5, 7, 9, 11, 13
  - Solve Logarithmic Equations: 334#17, 19, 21, 22, 23, 25, 27, 29
  - Mixed Review: 336#75, 77, 79, 83, 87

6-07 Modeling with Exponential and Logarithmic Functions (6.7)

After this lesson…

- I can use a common ratio to determine whether data can be represented by an exponential function.
- I can use technology to find exponential models and logarithmic models for sets of data.

6-07 Modeling with Exponential and Logarithmic Functions (6.7)

- Choosing Functions to Model Data
- For equally spaced *x*-values
  - If *y*-values have common ratio (multiple) → exponential

  - If *y*-values have finite differences → polynomial

6-07 Modeling with Exponential and Logarithmic Functions (6.7)

- Determine the type of function represented by each table.
- Example: 342#3

x	5	10	15	20	25	30
y	4	3	7	16	30	49

- Try 342#1

x	0	3	6	9	12	15
y	0.25	1	4	16	64	256

Finite differences

4 3 7 16 30 49

-1 4 9 14 19

5 5 5 5

2rd order differences are constant → quadratic

Common ratio r = 4 → exponential

6-07 Modeling with Exponential and Logarithmic Functions (6.7)

- Use the regression feature on a graphing calculator
  - TI-84
  - Enter points in STAT → EDIT
    - To see points go Y= and highlight Plot1 and press ENTER to keep it highlighted
    - Press Zoom and choose ZoomStat
  - Go to STAT → CALC → ExpReg for exponential OR LnReg for logarithmic

- NumWorks
- Choose Regression from homescreen
- In Data tab, enter points
- Go to Graph tab
  - To change regression type, press OK and choose a different regression
  - Read the answer off the bottom of the graph

6-07 Modeling with Exponential and Logarithmic Functions (6.7)

- Determine whether the data show an exponential relationship. Then write a function that models the data.
- Example 342#20

x	−3	−1	1	3	5
y	2	7	24	68	194

- Try 342#19

x	1	6	11	16	21
y	12	28	76	190	450

Use technology

$$y = 11.12(1.77)^x$$

$$y = 8.88(1.21)^x$$

6-07 Modeling with Exponential and Logarithmic Functions (6.7)

- Assignment: 15 total
  - Determine Type of Model: 342#1-4
  - Find Model from Table: 342#19, 20, 21, 22, 30, 31, 32
  - Mixed Review: 344#39, 41, 47, 49